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PRELIMINARY TEST SHRINKAGE ESTIMATOR BASED ON MMSE ESTIMATOR OF AVERAGE LIFE IN EXPONENTIAL DISTRIBUTION

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Summary

This paper presents a shrinkage estimator based on minimum mean square error estimator of average life of exponential distribution in type II censored data. A preliminary test is used to decide whether to use a one or two parameter exponential distribution in the given case. The bias and mean square error of the estimator thus obtained are discussed.

Key words : Shrinkage estimator, Exponential distribution, Order statistic, Chi-square distribution, Bias, Efficiency.

Introduction

In life testing an experiment is performed with n items. Since the items are costly, we cannot wait till all the items in the sample fail. Therefore, the process will be terminated when r items, at time $t_1, t_2, \ldots, t_r$ fail. The r observed times occur in order of magnitude forming a set of order static from the parent population. In this process life times will then be known exactly only for those items that fail by time t_r.

Let $\hat{\theta}$ be the MLE of θ, the average life. A modified estimator $\hat{\theta}_c$ of θ is obtained by multiplying θ by some constant C i.e. $\hat{\theta}_C = C\hat{\theta}$, where C is chosen such that MSE of $\hat{\theta}_C$ is minimum. In this case, $\hat{\theta}_C$ is called the minimum mean square error (MMSE) estimator of θ. Let θ_o be the prior value of θ. Thompson [8] proposed a technique of estimation by shrinking $\hat{\theta}_C$ towards θ_o in estimation space. The new estimator obtained by using $\hat{\theta}_C$ called the shirnkage estimator, is better than the MVUE near the natural origin θ_0.

The shrinkage estimator of θ is obtained by shrinking $(\hat{\theta}_c - \theta_0)$ towards natural origin, near zero, and multiplying it by K, i.e.

$$\hat{\theta}_s = K(\hat{\theta}_c - \theta_o) + \theta o$$

where $0 < K < 1$.

Generally in life testing either a one parameter or two parameter exponential distribution is used. The densities of these distributions are given below :

$$f_1(t) = \begin{cases} (1/\theta)\exp(-t/\theta) & 0 \le t < \infty \quad \text{(Model one)} \\ 0 & \text{Otherwise} \end{cases}$$

and

$$f_2(t) = \begin{cases} (1/\theta)\exp(-(t-A)/\theta) & A \le t < \infty \quad \text{(Model two)} \\ 0 & \text{Otherwise} \end{cases}$$

Here θ is average life and A is minimum guarantee.

In order to decide whether model one or model two will be most appropriate for given problem, we perform a preliminary test of significance (PTS) for testing the hypothesis $H_0 : A = 0$ against the alternative hypothesis $H_1 : A \ne 0$. When the null hypothesis is true we use the model one and when alternative hypothesis is true we use the model two.

The use of PTS was first made by Bancroft [1]. For the detailed bibliography on PTS, see Bancroft and Han [2]. For life time distribution the use of PTS has been made by Richards [6], Saxena and Gupta [7], Gupta and Singh [5] and Bhatkulikar [3].

Suppose that n units have been placed on test in an experiment and that r of these have failed at times designated by $t_1, t_2, \ldots, t_r$ with no replacement of failed items. The r observed times occur in order of magnitude forming a set of order statistics from the parent population.

Following two statistics $\hat{\theta}_{r,n}$ and $\hat{\hat{\theta}}_{r,n}$ are unbiased estimators of the parameter θ (average life) based on model one and model two respectively:

$$\hat{\theta}_{r,n} = \left\{ \sum_{i=1}^{r} t_i + (n-r)\, t_r \right\} \times \frac{1}{r} = \sum_{i=1}^{r} W_i / r \tag{1.1}$$

and

$$\hat{\hat{\theta}}_{r,n} = \left\{ \sum_{i=2}^{r} (t_i - t_1) + (n-r)(t_r - t_1) \right\} \times \frac{1}{(r-1)} = \sum_{i=2}^{r} W_i/(r-1) \tag{1.2}$$

where $W_i = (n - i + 1)(t_i - t_{i-1})$

We shall use the following results due to Epstien and Sobel [4]:

When the parent population is exponentially distributed the random variable $2W_i/\theta$ follows a chi-square distribution with two degrees of freedom for $i = 2, 3, \ldots, r$. It was also shown that W_i's are mutually independent. If A=O then $2nt_i/\theta$ also follows a chi- square distribution and is independent of all other W_i's. If $A \neq 0$ then $2n(t_i - A)/\theta$ is the random variable which has these properties.

Thus when A=O, $2r\hat{\theta}_{r,n}/\theta$ follows chi-square distribution with 2r degrees of freedom. When $A \neq 0$ then $2(r-1)\hat{\hat{\theta}}_{r,n}/\theta$ follows a chi- square distribution with 2r–2 degrees of freedom.

Therefore, we use the statistic to test H_o :

$$F = \frac{n(r-1)(t_i - A)}{\left\{\sum_{i=2}^{r} (t_i - t_1) + (n - r)(t_r - t_1)\right\}} = \frac{n(r-1)(t_i - A)}{\sum_{i=2}^{r} W_i} \tag{1.3}$$

which follows, under H_o, a central F distribution with 2 and 2r—2 degrees of freedom. The null hypothesis is to be rejected i.e. model two is used when $F \geq F(\alpha; 2, 2r-2)$. If $F < F(\alpha; 2, 2r-2)$ we prefer to use model one.

2. *Mathematical Formulation*

Three estimators $\hat{\theta}_{r,n}$, $\hat{\hat{\theta}}_{r,n}$ and $\theta'_{r,n}$ have been studied out of which the first two are MLE's. The mean square errors of $\hat{\theta}_{r,n}$ and $\hat{\hat{\theta}}_{r,n}$ are θ^2/r and $\theta^2/(r-1)$ respectively.

We now consider the modified estimator θ'_c of θ defined as follows:

$$\theta'_c = \hat{\hat{\theta}}_c I + \hat{\theta}_c (1 - I)$$

where $I = 0$ if $F < F(\alpha, 2, 2r-2)$ (i.e. Model one used),

$I = 1$ if $F \geq F(\alpha, 2, 2r-2)$ (i.e. Model two used)

and $\hat{\theta}_c = C_1 \hat{\theta}_{r,n}$, $\hat{\hat{\theta}}_c = C_2 \hat{\hat{\theta}}_{r,n}$

where $C_1 = r/(r+1)$ and $C_2 = (r-1)/r$

CASE I : When $I = 0$ i.e. $\theta'_c = \hat{\theta}_c$. The shrinkage estimator $\hat{\theta}_{s1}$ is given by

$$\hat{\theta}_{s1} = K_1 \hat{\theta}_c + (1 - K_1)\theta_0 \tag{2.1}$$

giving $E(\hat{\theta}_{s1}) = K_1 C_1 \theta + (1 - K_1) \theta_0$

The bias in $\hat{\theta}_{s1}$ expressed as a fraction of θ, called the relative bias (R.B.), is given by

$$\text{R.B.} (\hat{\theta}_{s1}) = (K_1 C_1 - 1) + (1 - K_1) (\theta_0/\theta) \tag{2.2}$$

and

$$\text{MSE}(\hat{\theta}_{s1}) = \frac{(K_1C_1\theta)^2}{r} + (1-K_1)^2 \theta_0^2 + (K_1C_1-1)^2 \theta^2 + 2(K_1C_1-1-K_1^2C_1+K_1) \theta_0\theta \tag{2.3}$$

The value of K1 which minimizes the $\text{MSE}(\hat{\theta}_{s1})$ is given by

$$K_1 = \frac{\theta_0^2 + C_1 \theta^2 - C_1 \theta_0 \theta - \theta_0 \theta}{\frac{C_1^2 \theta^2}{r} + \theta_0^2 + C_1^2 \theta^2 - 2 C_1 \theta_0 \theta}$$

K_1 depends on unknown parameter θ . An estimate of K_1 may be obtained by replacing θ by its MLE.

CASE II : When I = 1 i.e. $\hat{\theta}_c = \hat{\hat{\theta}}_c$. The shrinkage estimator $\hat{\hat{\theta}}_{s1}$ is given by

$$\hat{\theta}_{s1} = K_2 \hat{\theta}_2 + (1 - K_2) \theta_0 \tag{2.4}$$

giving

$$E(\hat{\hat{\theta}}_{s1}) = K_2 C_2 \theta + (1 - K_2) \theta_0$$

The relative bias in $\hat{\hat{\theta}}_{s1}$ is given by

$$\text{R.B.}(\hat{\hat{\theta}}_{s1}) = (K_2 C_2 - 1) + (1 - K_2) (\theta_0/\theta) \tag{2.5}$$

Also

$$\text{MSE}(\hat{\hat{\theta}}_{s1}) = (K_2C_2\theta)^2/(r - 1) + (1 - K_2)^2 \theta_0^2 + 2 (1 - K_2) (K_2C_2 - 1) \theta_0 \theta + (K_2C_2 - 1)^2 \theta^2 \tag{2.6}$$

The value of K_2 which minimizes the $\text{MSE}(\hat{\hat{\theta}}_{s1})$ is given by

$$K_2 = \frac{\theta_0^2 + C_2\,\theta^2 - C_2\,\theta_0\,\theta - \theta_0\,\theta}{(C_2\theta)^2/(r-1) + \theta_0^2 + C_2^2\,\theta^2 - 2C_2\,\theta_0\,\theta}$$

K_2 depends on unknown parameter θ. An estimate of K_2 may be obtained by replacing θ by its MLE.

CASE III : When I is neither 0 nor 1. Then $\hat{\theta}_c = C\hat{\theta}_{r,n}$.

Where $$C = \frac{1}{\left\{I^2/(r-1) + (1-I)^2/r + 1\right\}}$$

Then the shrinkage estimator $\hat{\theta}_{s1}$ is

$$\hat{\theta}_{s1} = K_3\,\hat{\theta}_c + (1 - K_3)\,\theta_0$$

Therefore,

$$E(\hat{\theta}_{s1}) = K_3 C\theta + (1 - K_3)\,\theta_0$$

and the relative bias in $\hat{\theta}_{s1}$ is given by

$$\text{R.B.}\ (\hat{\theta}_{s1}) = (K_3 C - 1) + (1 - K_3)\,(\theta_0/\theta)$$

Also, $$\text{MSE}(\hat{\theta}_{s1}) = \frac{(K_3 CI\theta)^2}{(r-1)} + \frac{\{K_3\,C\,(1-I)\,\theta\}^2}{r}$$

$$+ (1-K_3)^2\,\theta_0^2 + (K_3C-1)^2\theta^2 + 2(1-K_3)\,(K_3C-1)\,\theta_0\,\theta$$

The value of K_3 which minimizes the MSE $(\hat{\theta}_{s1})$ is given by

$$K_3 = \frac{\theta_0^2 + C\theta^2 - C\theta_0\,\theta - \theta_0\,\theta}{\dfrac{(CI\theta)^2}{(r-1)} + \dfrac{\{C(1-I)\theta\}^2}{r} + \theta_0^2 + C^2\,\theta^2 - 2C\theta_0\,\theta}$$

This again is a function of θ. An estimator K_3 is obtained by replacing θ by its MLE.

3. *Relative Bias and MSE of Preliminary Test Shrinkage Estimator of MMSE Estimator*

Epstien and Sobel [4] have shown that the quantities U_1, U_2 and U_3 given below are independently distributed as central chi-square with 2, 2r-2 and 2r degrees of freedom respectively.

Where $U_1 = 2n\,(t_1 - A)/\theta$

$$U_2 = 2\Big\{ \sum_{i=2}^{r} (t_i - t_1) + (n - r)(t_r - t_1) \Big\} \times \frac{1}{\theta}$$

and $$U_3 = 2\left\{ \sum_{i=1}^{r} t_i + (n - r)\, t_r \right\} \times \frac{1}{\theta}$$

The joint p.d.f. of U_1, U_2 and U_3 is given by

$$g(U_1, U_2, U_3) = \{2^{2r}\, \Gamma r \Gamma(r-1)\}^{-1} \exp\{-(\tfrac{1}{2})(U_1 + U_2 + U_3)\}\, U_2^{r-2}\, U_3^{r-1}$$

Using the following transformations

$$F = AU_1/(VU_2),\quad U_2 = U_2 \quad \text{and} \quad U_3 = U_3$$

where $A = \tfrac{1}{2}$ and $V = 1/\{2(r-1)\}$.

The joint destiny of F, U2 and U3 is given by

$$g_1(F, U_2, U_3) = \{2^{2r}(\Gamma r)^2\}^{-1} \exp\{-(FVU_2 + AU_2 + AU_3)\}\, U_2^{r-1}\, U_3^{r-1}$$

Allowing a preliminary test to determine the model to be used, it is necessary to compute $E(\theta_{s1})$ and $E(\theta_{s1})^2$

$$\theta_{s1} = \hat{\hat{\theta}}_{s1}\, I + (1 - I)\, \hat{\theta}_{s1}$$

Consider first

$$E(\theta_{s1}) = E(\hat{\theta}_{s1} \mid F < F_\alpha)\Pr(F < F_\alpha) + E(\hat{\hat{\theta}}_{s1} \mid F > F_\alpha)\Pr(F > F_\alpha)$$

$$= E_1 + E_2 \tag{3.1}$$

where $$E_1 = \int_{U_3=0}^{\infty} \int_{U_2=0}^{\infty} \int_{F=0}^{F_\alpha} \hat{\theta}_{s1}\, g_1(F, U_2, U_3)\, dF.dU_2.dU_3 \tag{3.2}$$

and $$E_2 = \int_{U_3=0}^{\infty} \int_{U_2=0}^{\infty} \int_{F=F_\alpha}^{\infty} \hat{\hat{\theta}}_{s1}\, g_1(F, U_2, U_3)\, dF.dU_2.dU_3 \tag{3.3}$$

Putting the value of $\hat{\theta}_{s1}$ and $\hat{\hat{\theta}}_{s1}$ from equations (2.1) and (2.4) in (3.2) and (3.3), making suitable transformations and simplifying the integral, we finally get from (3.1):

$$E(\theta_{s1}) = \{1 - (1 + VF_\alpha/A)^{-r+1}\}\{K_1 C_1 \theta + (1-K_1)\theta_0\}$$

$$+ (1 + VF_\alpha/A)^{-r} \left\{K_2 C_2 \theta + (1 - K_2)\theta_0 (1 + VF_\alpha/A)\right\} \quad (3.4)$$

The bias in $\hat{\theta}_{s1}$ as a fraction of q is then given by

$$\text{R.B.}(\hat{\theta}_{s1}) = \{ \{1 - (1+VF_\alpha/A)^{-r+1} \} \{ K_1C_1 \theta + (1 - K_1) \theta_0 \}$$

$$+ (1 + VF_\alpha/A)^{-r} \{K_2C_2 \theta + (1-K_2)\theta_0 (1 + VF_\alpha/A) \} \} \times \frac{1}{\theta - 1}$$

$$(3.5)$$

To calculate the mean square error of $\hat{\theta}_{s1}$, we first calculate $E(\hat{\theta}_{s1})^2$. Using calculus, we get :

$$\text{MSE}(\hat{\theta}_{s1}) = \left\{1-(1+VF_\alpha/A)^{-r+1}\right\} \left\{K_1^2C_1^2 \theta^2 (r+1)/r + (1-K_1)^2\theta_0^2 + 2K_1C_1 (1-K_1) \theta_0 \theta\right\}$$

$$+ (1+VF_\alpha/A)^{-r-1} \left\{K_2^2 C_2^2\theta^2 r/(r-1) + (1 - K_2)^2\theta_0^2 (1 + VF_\alpha/A)^2\, 2(1 - K_2) K_2C_2\theta_0\, \theta(1+VF_\alpha/A)\right\}$$

$$-\theta\left\{2\left\{1- (1+VF_\alpha/A)^{-r+1}\right\} \left\{K_1C_1\theta + (1-K_1)\theta_0\right\} +2(1+VF_\alpha/A)^{-r} \left\{K_2C_2 \theta + (1-K_2)\theta_0 (1+VF_\alpha/A)\right\} - \theta\right\}$$

$$(3.6)$$

4. *Mathematical Results*

Some results are given in the form of theorems regarding to the behaviour of bias and mean square error of the preliminary test modified shrinkage estimator $\hat{\theta}_{s1}$

Theorem 4.1 : For given value of r and n the bias in $\hat{\theta}_{s1}$ will be always minimum algebrically for

$$F_\alpha = \frac{r\theta K_2C_2 - (r - 1)(K_1C_1\theta - K_1\theta_0 + K_2\theta_0)}{(K_1C_1\theta - K_1\theta_0 + K_2\theta_0)} \quad (4.1)$$

Proof: From equation (3.4), bias in $(\hat{\theta}_{s1})$ is given by

$$\text{Bias}(\hat{\theta}_{s1}) = \left\{1 - (1 + VF_\alpha/A)^{-r+1}\right\} \left\{K_1C_1\theta + (1 - K_1) \theta_0\right\}$$

$$+ (1+VF_\alpha/A)^{-r} \left\{K_2C_2 \theta + (1-K_2) \theta_0 (1+VF_\alpha/A)\right\} - \theta \quad (4.2)$$

Differentiating equation (4.2) with respect to F_α and equating it to zero, we get two values of F_α. The first value is (-A/V) which is negative. Since F_α cannot be negative, we consider the other value given by (4.1). Differentiating twice (4.2) with respect to F_α and putting in it the value of F_α given by (4.1), we get

$$\delta^2 \text{ Bias } (\hat{\theta}_{s1})/\delta F_\alpha^2 = \left\{rK_2C_2\theta/(r-1)^2\right\} (1+VF_\alpha/A)^{-r-2}$$

The right hand side is positive and the theorem is proved.

Theorem 4.2 : For given value of r and n and for $F_\alpha \to \infty$ the bias and mean square error of $\hat{\theta}_{s1}$ will be equal to the bias and mean square error of $\hat{\theta}_{s1}$

Theorem 4.3 : For given value of r and n and $F_\alpha = 0$ the bias and mean square error of $\hat{\theta}_{s1}$ will be equal to the bias and mean square error of $\hat{\hat{\theta}}_{s1}$

Proof of theorem 4.2 and 4.3 are obvious from equation (3.4) and (3.6).

5. *Discussion of Results*

The bias and mean square error of the preliminary test modified shrinkage estimator $\hat{\theta}_{s1}$ are function of r, θ_0/θ and level of significance of preliminary test α. For a given experiment r is the number of observations that failed at time $t_1, t_2, \ldots, t_r$ and is prefixed. The parameter θ is unknown. Hence the only parameter at our disposal is α. We plan to select a suitable value of α which will minimize the bias and mean square error of $\hat{\theta}_{s1}$

For this purpose an empirical study has been made for two different value of r (4,10), for different value of θ_0/θ (.5, .7, 1.1, 1.3, 1.5) and five different values of α (0.00, 0.01, 0.05, 0.25 and 1.00).

The statistics $\hat{\theta}_{r,n}$ and $\hat{\hat{\theta}}_{r,n}$ are the MLE's of θ. The MSE's of these estimators are θ^2/r and $\theta^2/(r-1)$ The relative efficiencies of $\hat{\theta}_{s1}$ with respect to $\hat{\theta}_{r,n}$ and $\hat{\hat{\theta}}_{r,n}$ are defined by

$$RE_I\ (\hat{\theta}_{s1} \text{ w.r.to } \hat{\theta}_{r,n}) = \frac{MSE(\hat{\theta}_{r,n})}{MSE(\hat{\theta}_{s1})} \times 100\%$$

$$RE_{II}\ (\hat{\theta}_{s1} \text{ w.r.to } \hat{\hat{\theta}}_{r,n}) = \frac{MSE(\hat{\hat{\theta}}_{r,n})}{MSE(\hat{\theta}_{s1})} \times 100\%$$

The values of relative bias of $\hat{\theta}_{s1}$ are summarised in table 1 and those of relative efficiency in table 2.

From table 1, we observe that the bias decreases as r increases, as it should be. We also see that for $\theta_0/\theta < 1$ the bias is minimum and almost constant for $\alpha \le .05$ and for $\theta_0/\theta > 1$ the bias is minimum at $\alpha = .25$. Hence from the table of bias we conclude that it is minimum at $\alpha = .05$ when $\theta_0/\theta < 1$ and at $\alpha = 0.25$ when $\theta_0/\theta > 1$

From table 2, the preliminary test modified shrinkage estimator $\hat{\theta}_{s1}$ is always more efficient than $\hat{\theta}_{r,n}$ and $\hat{\hat{\theta}}_{r,n}$ both. The relative efficiency is also greater than $\hat{\theta}_C$ For $\theta_0/\theta < 1$ the efficiency for $\alpha \leq .05$ is greater than that for $\alpha > .05$ and for $\theta_0/\theta > 1$ the efficiency is maximum at $\alpha = .25$. We therefore recommend that

(i) for $\theta_0/\theta < 1$, use $\alpha = .05$ as the level of PTS, and

(ii) for $\theta_0/\theta > 1$, use $\alpha = .25$ as the level of PTS.

The estimator thus obtained, though biased, will be more efficient than the MLE and modified estimator $\hat{\theta}_C$

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APPENDIX

Table 1 : Relative bias of preliminary test shrinkage estimator θ_{s1} of MMSE estimator.

θ_0/θ	r	Level of significance of preliminary test				
		0.00	0.01	0.05	0.25	1.00
0.5	4	-0.320	-0.320	-0.321	-0.347	-0.375
	10	-0.165	-0.165	-0.165	-0.177	-0.180
0.7	4	-0.282	-0.282	-0.282	-0.287	0.296
	10	-0.196	-0.196	-0.196	-0.204	-0.208
1.1	4	0.064	0.064	0.064	0.059	0.060
	10	0.069	0.069	0.069	0.067	0.069
1.3	4	0.117	0.117	0.117	0.103	0.115
	10	0.105	0.105	0.105	0.099	0.108
1.5	4	0.123	0.123	0.122	0.102	0.125
	10	0.096	0.096	0.096	0.087	0.100

Table 2 : Relative efficiency of preliminary test shrinkage estimator θ_{s1} of MMSE estimator with respect to $\hat{\theta}_{r,n}$ and $\hat{\hat{\theta}}_{r,n}$

θ_0/θ	r	Level of significance of preliminary test				
		0.00	0.01	0.05	0.25	1.00
0.5	4	156(208)	156(208)	155(207)	140(186)	133(178)
	10	122(135)	122(135)	122(135)	114(126)	111(123)
0.7	4	298(397)	298(397)	298(397)	288(384)	282(376)
	10	172(192)	172(192)	172(192)	156(174)	159(176)
1.1	4	3906(5208)	3906(5208)	4167(5555)	4545(6061)	3571(4762)
	10	1250(1389)	1250(1389)	1250(1389)	1429(1587)	1429(1587)
1.3	4	708(944)	708(944)	731(975)	789(1051)	733(977)
	10	312(347)	312(347)	312(347)	333(370)	312(347)
1.5	4	406(541)	406(541)	403(538)	439(586)	400(533)
	10	213(236)	213(236)	213(236)	217(241)	200(222)

Note : The values of REII, i.e., R.E. of θ_{s1} w.r. to $\hat{\hat{\theta}}_{r,n}$ are shown in brackets.